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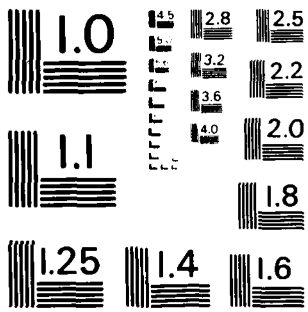
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TURBULENT BOUNDARY LAYERS OVER ROUGH SURFACES IN HYPRSONIC FLOW

by

John M. Russell

Abstract

A method for predicting the downstream development of momentum thickness, skin friction, and heat transfer in a supersonic turbulent boundary layer over a rough flat plate based on ideas of Van Driest, Rotta, and Bradshaw is derived and discussed. *Admissable thermal boundary conditions include the case of prescribed* wall temperature and the case of an adiabatic wall. The velocity profiles for compressible nonadiabatic flow are expressed as transformations of the corresponding velocity profiles in incompressible adiabatic flow. Analytical curve fits to the experimentally determined law-of-the-wall (including the sublayer region) are given, as are analytical representations of the effects of sand grain roughness based on the well known data of Nikuradse.

A FORTRAN source code for implementing the method is included as are sample calculations of the momentum thickness, skin friction, and heat transfer for several roughness heights.

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CONTENTS

I. INTRODUCTION

Interest in the feasibility and technical merit of a high-altitude high speed cruise missile has inspired recent Air Force interest in turbulent boundary layers over rough surfaces. Apart from its specific military applications, the problem is of fundamental scientific and engineering interest. Though the subject is at least as old as the supersonic flight of aircraft, the physical mechanisms controlling the main qualitative features of such flows are still largely a matter of debate, and there is ample room for improvements in our level of understanding of the physics of the flows and the equations used to describe them.

A long-range experimental program under the direction of Dr. Anthony W. Fiore at the High Speed Aero Performance Branch, Wright Aeronautical Laboratories, Dayton has been underway for over two years. Among the goals of that program are the aquisition of quality experimental data contrasting the entropy producing mechanisms of various milled surface roughness types on a flat plate in Mach 6 flow. Sometimes the interpretation of experimental data is assisted by comparison with numerical predictions, and the furnishing of such predictions was the original purpose of the present investigations.

As it happens, we may have achieved a modest simplification and improvment of the existing technology of numerical prediction of flat plate boundary layers. For example, the method presented herein requires only that an explicit first order ordinary differential equation be solved for the downstream develoment of the skin friction. The momentum thickness is expressed as an explicit function of the skin friction (provided the parameters of the free stream flow, the thermal boundary conditions, and the roughness height are specified).

II. OBJECTIVES

Our objectives, as already stated, are

(i) To provide an efficient and scientifically plausible algorithm for the calculation of the downstream development of the momentum thickness, skin friction parameter and heat transfer parameter for realistic input data.

(ii) To develop a FORTRAN source code to implement the algorithm.

(iii) To carry out sample calculations that forecast the outcome of experiments underway at AFWAL

III. A BOUNDARY LAYER CALCULATION ALGORITHM

In this section, we will show, starting with the equations of the motion of a visions compressible fluid, that the downstream development of a turbulent boundary layer over a rough surface in supersonic flow can be modeled effectively by a single first order differential equation for a certain friction parameter. The method employs a few well known results, a few less well known results, and a few curve fits to experimental data which appear here for the first time. We begin by deriving an energy equation due to Rotta (1959, 1960) which then provides the basis for a compressible law-of-the-wall which is a slight generalization of the well known Van Driest transformation (Van Driest, 1951).

A. Rotta's Energy Equation

Let (x_1,x_2x_3) denote a Cartesian coordinate system with x_1 the streamwise coordinate, x_2 the vertical coordinate normal to the wall and x_3 the spanwise coordinate defined positive in the right-handed sense. Let (u_1, u_2, u_3) be the local instantaneous velocity field. Let p, ρ, and T be the instantaneous pressure, mass density and temperature, respectively. Let $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ be the local heat flux vector due to thermal conduction. Let $\mu(T)$ denote the molecular viscosity and let k(T) denote the thermal conductivity. Let σ_{kj} (j = 1,2,3; k = 1,2,3,) denote the set of components of the state of stress tensor. Let c_p and c_v denote the specific heats of constant pressure and at constant volume respectively. Let $R = c_p - c_v$. Then the equation of state for an ideal gas reads

$$p_e = \rho RT$$

where the subscript "e" signifies equilibrium pressure to distinguish it from the local instantaneous mechanical pressure p defined by

$$p = - \frac{\sigma_{kk}}{3}$$

(Here, as elsewhere, the summation convention is understood.) These two types of pressure are related by

$$p - p_e = - \mu_B \frac{\partial u_j}{\partial x_j}$$

where μ_B is the bulk viscosity. Let

$$e_{kj} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right)$$

denote the elements of the rate-of-deformation tensor. Then the Stokes viscosity formula reads

$$\sigma_{kj} = -p\, \delta_{kj} + 2\mu \left[e_{kj} - \frac{e_{mm}}{3} \delta_{kj}\right].$$

The Fourier heat condition law reads

$$\dot{q}_k = -k \frac{\partial T}{\partial x_k}.$$

Let e denote the thermometric internal energy. Let R denote a simply connected region in space and S its bounding surface. Let dV and $d\Sigma$ denote the differential volume element and the differential area element within R and on S, respectively. Finally, let (n_1, n_2, n_3) denote the components of the outward unit normal vector on S. Then the "control volume" forms of the equations of conservation of mass momentum, and energy are, respectively,

$$\frac{d}{dt} \left(\iiint_R \rho dV\right) = - \iint_S (\rho u_k) n_k \, d\Sigma$$

$$\frac{d}{dt} \left(\iiint_R \rho\, u_j \, dV\right) = - \iint_S (\rho u_j) u_k n_k d\Sigma + \iint_S \sigma_{jk} n_k d\Sigma$$

$$\frac{d}{dt} \left(\iiint_R \rho\left(e + \frac{u_j u_j}{2}\right) dV\right) = - \iint_S \rho u_k \left(e + \frac{u_j u_j}{2}\right) n_k \, d\Sigma$$

$$+ \iint_S \sigma_{kj} n_k u_j \, d\Sigma - \iint_S \dot{q}_k n_k \, d\Sigma$$

provided we ignore body forces such as gravity.

Let $\tau_{kj} \equiv 2\mu(e_{kj} - \frac{e_{mm}}{3}\delta_{kj})$ denote the anisotropic or shear stress part of the σ_{kj} matrix. Then in view of previous definitions, the Stokes viscosity formula can be written

$$\sigma_{kj} = - p_e \delta_{kj} + \mu_B \frac{\partial u_m}{\partial x_m} \delta_{kj} + \tau_{kj} .$$

Let the ensemble average operator be denoted by pointed brackets $\langle \ \rangle$. Assuming the flow is statistically stationary, the energy equation becomes

$$0 = - \iint_S \langle \rho u_k (e + \frac{p_e}{\rho} + \frac{u_j u_j}{2}) \rangle n_k \, d\Sigma$$

$$+ \iint_S \langle (\mu_B \frac{\partial u_m}{\partial x_m} \delta_{kj} + \tau_{kj}) u_j \rangle n_k \, d\Sigma - \iint_S \langle \dot{q}_k \rangle n_k \, d\Sigma .$$

The quantities in the first line of this equation represent "bulk mixing" effects, namely transport of internal energy, reversible pressure work done on the fluid in the control volume by the surroundings, and transport of kinetic energy, respectively.

The quantities in the second line, by contrast, represent "molecular mixing" effects. If the Reynolds number is large and the Prandtl number of the fluid is of order unity (as it is for air) then the "molecular mixing" effects will be important only in the sublayer region near the wall. Under these same hypotheses the "bulk mixing effects" will be dominant in the interior of the flow away from the wall.

Let $(u,v,w) = (u_1,u_2,u_3)$ and $(x,y,z) = (x_1,x_2,x_3)$. Also let $\{\hat{i},\hat{j},\hat{k}\}$ denote the right handed orthonormal triad of basis vectors associated with the (x,y,z) coordinate system. We now consider a specific choice of the control volume R.

Let R be a cross sectional fluid slab of thickness dx in the x-direction, of height y in the y direction, and of span b in the z-direction. Assuming air is "calorically perfect," we have

$$e = c_v T$$

Thus, with

$$e + \frac{p_e}{\rho} = c_p T.$$

Then on the top face of the slab (where $\hat{n} = \hat{j}$), the bulk mixing effects dominate over the molecular mixing effects so that

$$\langle \rho v (e + \frac{p_e}{\rho}) \rangle = c_p \langle \rho v T \rangle >> - \langle \dot{q}_2 \rangle$$

and

$$\langle \rho v \frac{u_j u_j}{2} \rangle >> \langle \mu_B \frac{\partial u_m}{\partial x_m} \delta_{2j} + \tau_{2j}) u_j \rangle .$$

We will assume that the bottom surface of the slab ($\hat{n} = - \hat{j}$) lies on the wall at the level $y = 0$. Also the wall is rigid and impermeable to mass. We also assume that the flow is statistically homogeneous in the z-direction so that the net energy flux across the faces with $\hat{n} = \hat{k}$ and $\hat{n} = - \hat{k}$ is zero. Then the energy equation becomes

$$0 = - c_p \langle \rho v T \rangle - \langle \rho \frac{u_j u_j}{2} v \rangle$$

$$+ \frac{\partial}{\partial x} \left[\int_o^y \left(-c_p \langle \rho u T \rangle - \langle \rho u \frac{u_j u_j}{2} \rangle + \langle \mu_B \frac{\partial u_m}{\partial x_m} u \rangle \right. \right.$$

$$\left. \left. + \langle \tau_{1j} u_j \rangle - \langle \dot{q}_1 \rangle \right) dy' \right]$$

$$+ \langle \dot{q}_2 \rangle_{y=0} .$$

The quantity involving the x-derivative represents the net contribution to the energy flux across the forward and backward faces of the slab. The quantities in the first and last line represent the dominant energy flux terms across the top and bottom faces of the slab, respectively.

In the "wall region" of the flow, all mean flow quantities are functions of the nondimensional variable y^+ defined by

$$y^+ = \frac{y u_\tau}{\nu_w} \equiv \frac{y}{l_{visc}}$$

where ν_w is the wall value of the kinematic viscosity and u_τ is the friction velocity. For boundary layer flows in general and for flat plate flow in particular, l_{visc} is slowly varying in x, i.e.

$$\frac{dl_{visc}}{dx} << 1$$

We will accordingly, neglect the x-derivative terms in the energy equation and obtain the leading approximation.

$$-c_p\langle\rho vT\rangle = \langle\rho \frac{u_j u_j}{2} v\rangle + \dot{q}_w$$

where

$$\dot{q}_w \equiv \langle\dot{\vec{q}} \cdot \hat{n}\rangle_{y=0} = - \langle\dot{q}_2\rangle_{y=0}$$

The next step in deriving Rotta's energy equation is to apply the equations of conservation of momentum and of mass in order to rewrite the kinetic energy transport term in a more useful form.

Assuming, as before, that the flow is statistically stationary the ensemble average of the momentum and mass equations become

$$0 = - \iint_S \langle\rho u_j u_k\rangle n_k \, d\Sigma - \iint_S \langle p_e\rangle n_j d\Sigma + \iint_S \langle\mu_B \frac{\partial u_m}{\partial x_m} \delta_{kj} + \tau_{kj}\rangle n_k d\Sigma$$

and

$$0 = - \iint_S \langle\rho u_k\rangle n_k \, d\Sigma \ ,$$

respectively.

On the top face $(\hat{n} = \hat{j})$ of S , the "bulk mixing" terms dominate over the "molecular mixing" terms, so that

$$\langle\rho u_j v\rangle >> \langle\mu_B \frac{\partial u_m}{\partial x_m} \delta_{2j} + \tau_{2j}\rangle$$

Assuming the flow is statistically homogeneous in z as before, the momentum flux balance and the mass flux balance for a cross sectional fluid slab become

$$0 = -\langle \rho u_j v\rangle - \langle p_e\rangle\delta_{2j}$$
$$+ \frac{\partial}{\partial x}\left[\int_o^y \left(-\langle \rho u_j u\rangle - \langle p_e\rangle\delta_{1j} + \langle \mu_B \frac{\partial u_m}{\partial x_m}\delta_{1j} + \tau_{1j}\rangle\right)dy'\right]$$
$$+ \langle p_e\rangle_{y=0}\delta_{2j} - \langle \mu_B \frac{\partial u_m}{\partial x_m}\delta_{2j} + \tau_{2j}\rangle_{y=0}$$

and

$$0 = -\langle \rho v\rangle - \frac{\partial}{\partial x}\left[\int_o^y \langle \rho u\rangle\, dy'\right] ,$$

respectively. We assume, as before, that the mean flow quantities (except the pressure) are functions of $y^+ = y/l_{visc}$ only and that

$$\frac{dl_{visc}}{dx} << 1$$

Also, for flat plate flow

$$\langle p_e\rangle \equiv P = \text{constant}.$$

It follows that the x-derivative terms in the above equations are zero. Also, from the no-slip boundary condition on y = 0 we have

$$\langle \mu_B \frac{\partial u}{\partial x}\rangle_{y=0} = \langle \mu_B \frac{\partial w}{\partial z}\rangle_{y=0} = 0$$

Letting

$$\langle \tau_{21}\rangle_{y=0} \equiv \tau_w,$$

the momentum and mass conservation equations become

$$0 = \langle \rho u_j v\rangle + \langle \mu_B \frac{\partial v}{\partial y}\rangle_{y=0}\,\delta_{2j} + \tau_w\,\delta_{1j}$$

and

$$0 = \langle \rho v \rangle,$$

respectively. We may now use these results to simplify the energy equation

$$-c_p\langle \rho v T\rangle = \langle \rho \frac{u_j u_j}{2} v\rangle + \dot{q}_w$$

derived earlier.

Let $U_j = \langle u_j \rangle$

$u_j' = u_j - U_j$

Then $\frac{u_j u_j}{2} = \frac{U_j U_j}{2} + U_j u_j' + \frac{u_j' u_j'}{2}$

so that

$$\langle \rho \frac{u_j u_j}{2} v\rangle = \frac{U_j U_j}{2}\langle \rho v\rangle + U_j \langle \rho u_j' v\rangle + \langle \rho \frac{u_j' u_j'}{2} v\rangle$$

In view of the mass conservation equation $\langle \rho v \rangle = 0$, the first term on the right vanishes. Employing both mass and momentum, the second term becomes

$$\begin{aligned} U_j\langle \rho u_j' v\rangle &= U_j\langle \rho u_j v\rangle - U_j U_j \langle \rho v\rangle \\ &= U_j\langle \rho u_j v\rangle \\ &= -\, V\langle \mu_B \frac{\partial v}{\partial y}\rangle_{y=0} - U\tau_w \end{aligned}$$

where $U \equiv U_1$ and $V \equiv V_1$. In view of the ordinary boundary layer approximation, $V \ll U$, so the second term on the right dominates over the first. The kinetic energy transport term has been reduced to

$$\langle \rho \frac{u_j u_j}{2} v\rangle = -\, U\tau_w + \langle \rho \frac{u_j' u_j'}{2} v\rangle$$

We will neglect the second term on the right compared to the first since we expect third order products of fluctuations to be small compared to second order ones. With this approximation the full energy equation becomes

$$-c_p \langle \rho v T \rangle = -U\tau_w + \dot{q}_w$$

which we will call **Rotta's energy equation** (cf. Rotta (1959), equation 31).

B. Van Driest Transformation Theory

We now introduce the "turbulent Prandtl number" σ_t defined by

$$\sigma_t = \left(\frac{-\langle \rho u'v' \rangle}{\frac{dU}{dy}}\right)\left(\frac{\frac{d\bar{T}}{dy}}{-\langle \rho v T \rangle}\right)$$

where $\bar{T} \equiv \langle T \rangle$. We will assume that σ_t is a constant. Experimental results suggest $\sigma_t = 0.9$ is a sensible average according to Rotta (1960).

From the last form of the momentum equation written down in the last section (considering the $j = 1$ component) we have

$$-\langle \rho u'v' \rangle = \tau_w$$

(since $v' \approx v$). It follows from the above definition of σ_t that

$$-\langle \rho v T \rangle = \frac{d\bar{T}}{dU} \frac{\tau_w}{\sigma_t}$$

Rotta's energy equation then becomes

$$\frac{c_p \tau_w}{\sigma_t} \frac{d\bar{T}}{dU} = -U\tau_w + \dot{q}_w$$

Letting
$$\rho_w = \langle \rho \rangle_{y=0}$$
$$T_w = \langle T \rangle_{y=0}$$
$$\tau_w = \rho_w u_\tau^2$$

we have

$$\frac{d\left(\frac{\bar{T}}{T_w}\right)}{d\left(\frac{U}{u_\tau}\right)} = \left(\frac{\dot{q}_w}{\rho_w u_\tau T_w c_p}\right) \sigma_t - \frac{u_\tau^2 \sigma_t}{T_w c_p} \frac{U}{u_\tau}$$

Let
$$\beta_q \equiv \frac{\dot{q}_w}{\rho_w u_\tau T_w c_p}$$

$$a_w^2 \equiv \frac{c_p}{c_v}(c_p - c_v)T_w = c_p(\gamma-1)T_w$$

$$M_\tau \equiv \frac{u_\tau}{a_w}$$

Then
$$\frac{u_\tau^2 \sigma_t}{T_w c_p} = \frac{M_\tau^2 a_w^2 \sigma_t}{[a_w^2/(\gamma-1)]} = M_\tau^2(\gamma-1)\sigma_t$$

so that
$$\frac{d(\frac{\bar{T}}{T_w})}{d(\frac{U}{u_\tau})} = \beta_q \sigma_t - M_\tau^2(\gamma-1)\sigma_t(\frac{U}{u_\tau}) ,$$

hence
$$\frac{\bar{T}}{T_w} = C_1 + \beta_q \sigma_t (\frac{U}{u_\tau}) - \frac{M_\tau^2}{2}(\gamma-1)\sigma_t(\frac{U}{u_\tau})^2$$
$$= \frac{(\gamma-1)}{2} M_\tau^2 \sigma_t \{[(\frac{\beta_q}{(\gamma-1)M_\tau^2})^2 + \frac{C_1}{\frac{(\gamma-1)}{2}M_\tau^2\sigma_t}] - [\frac{U}{u_\tau} - \frac{\beta_q}{(\gamma-1)M_\tau^2}]^2\}$$

It is convenient to introduce a new variable ϕ defined by

$$\frac{U}{u_\tau} - \frac{\beta_q}{(\gamma-1)M_\tau^2} \equiv [(\frac{\beta_q}{(\gamma-1)M_\tau^2})^2 + \frac{C_1}{\frac{(\gamma-1)}{2}M_\tau^2\sigma_t}]^{1/2} \sin\phi \qquad (1)$$

Then, in terms of ϕ,

$$\frac{\bar{T}}{T_w} = (C_1 + \frac{\beta_q^2\sigma_t}{2(\gamma-1)M_\tau^2}) \cos^2\phi \qquad (2)$$

The above formula is very convenient for deriving a compressible flow generalization of the logarithmic form of the law of the wall.

Let δ denote the overall thickness of a turbulent boundary layer (from the wall to the average elevation of the turbulent-non-turbulent interface). We will refer to the "generalized logarithmic region" of the turbulent boundary layer as the range of values of y which satisfy, say,

$$(35)\ l_{visc} < y < (0.2)\ \delta.$$

Let the symbol κ denote the Kármán constant. Then the formula

$$\frac{dU}{dy} = \frac{1}{\kappa y}\left(\frac{\tau_w}{\bar{\rho}(y)}\right)^{1/2}$$

is well established for uniform denisty flows. Following Van Driest (1951), we will assume that this formula also holds for nonuniform mean density.

We have already employed the hypothesis that the mean pressure is constant (and uniform in space) in flat plate flow. From the equation of state for an ideal gas, we have, therefore,

$$P = (c_p - c_v)\bar{\rho}\bar{T}$$

$$P = (c_p - c_v)\bar{\rho}_w\bar{T}_w$$

(where the subscript "w" denotes the wall level (y=0). It follows that

$$\frac{\rho_w}{\bar{\rho}} = \frac{\bar{T}}{T_w}$$

an identity which we will employ frequently in what follows. With the definition $\tau_w = \bar{\rho}_w u_\tau^{\ 2}$, the formula for dU/dy becomes

$$\frac{dU}{dy} = \frac{u_\tau}{\kappa y}\left(\frac{\bar{T}(y)}{T_w}\right)^{1/2}$$

or

$$\frac{1}{\kappa}\frac{dy}{y} = \frac{d\left(\frac{U}{u_\tau}\right)}{\left(\frac{\bar{T}(y)}{T_w}\right)^{1/2}}$$

Our original definition (1) of the variable ϕ above was a formula relating U/u_τ to ϕ. Rotta's energy equation and constancy of σ_t then led to formula (2) which relates $\bar{T}/T_w$ to ϕ. It follows that (1) and (2) can be employed to eliminate U/u_τ

and $\bar{T}/T_w$ from the right hand side of the above equation. The result is found to be

$$\frac{1}{\kappa}\frac{dy}{y} = \frac{1}{\left(\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t\right)^{1/2}}\,d\phi$$

from which

$$\phi - \phi_o = \left(\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ln y^+ + C_2\right) \qquad (3)$$

(where we have introduced two constants of integration in order to permit one to be chosen for analytical convenience later).

From the identity

$$\sin\phi = \sin(\phi - \phi_o + \phi_o)$$

$$= \sin(\phi - \phi_o)\cos\phi_o + \cos(\phi - \phi_o)\sin\phi_o$$

Substituting this into (1) and using (3) to eliminate $\phi - \phi_o$, we have

$$\frac{U}{u_\tau} - \frac{\beta_q}{(\gamma-1)M_\tau^{\,2}} = \left[\left(\frac{\beta_q}{(\gamma-1)M_\tau^{\,2}}\right) - \frac{C_1}{\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t}\right]^{1/2}$$

$$\cdot\left\{\sin\left[\left(\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ln y^+ + C_2\right)\right]\cos\phi_o\right.$$

$$\left.+\cos\left[\left(\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ln y^+ + C_2\right)\right]\sin\phi_o\right\} \qquad (4)$$

Since both of the constants ϕ_o and C_2 arose from a single integration, we may choose a particular value for either one without loss of generality. We will choose ϕ_o such that

$$\lim_{\substack{\beta_q \to 0\\ M_\tau \to 0}}\left\{\frac{U}{u_\tau}\right\} = \frac{1}{\kappa}\ln y^+ + C_2$$

which will ensure that our formulas (which hold for compressible flow with heat transfer) reduce to the appropriate form in the incompressible adiabatic limit.

If we write (4) in the form

$$\frac{U}{u_\tau} = \left[\frac{\beta_q{}^2\sigma_t}{2(\gamma-1)M_\tau{}^2} + C_1\right]^{1/2}$$

$$\cdot \Big\{\left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{-1/2} \sin \left[\left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ \ln y^+ + C_2\right)\right] \cos \phi_o$$

$$+ \left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{-1/2} \cos \left[\left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ \ln y^+ + C_2\right)\right] \sin \phi_o$$

$$+ \frac{\beta_q}{(\gamma-1)M_\tau{}^2}\left[\frac{\beta_q{}^2\sigma_t}{2(\gamma-1)M_\tau{}^2} + C_1\right]^{-1/2}\Big\}$$

The above limit condition therefore leads to the choice

$$\cos \phi_o = (C_1)^{1/2}\left[\frac{\beta_q{}^2\sigma_t}{2(\gamma-1)M_\tau{}^2} + C_1\right]^{-1/2}$$

$$\sin \phi_o = \left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{1/2}\left(\frac{-\beta_q}{(\gamma-1)M_\tau{}^2}\right)\left[\frac{\beta_q{}^2\sigma_t}{2(\gamma-1)M_\tau{}^2} + C_1\right]^{-1/2}$$

(which incidentally, satisfies $\sin^2\phi_o + \cos^2\phi_o = 1$)

Eliminating ϕ_o from the above formula for U/u_τ, we get

$$\frac{U}{u_\tau} = \frac{\beta_q}{(\gamma-1)M_\tau{}^2} + C_1{}^{1/2}\left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{-1/2} \sin \left[\left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ \ln y^+ + C_2\right)\right]$$
$$- \frac{\beta_q}{(\gamma-1)M_\tau{}^2} \cos \left[\left(\frac{(\gamma-1)}{2} M_\tau{}^2\sigma_t\right)^{1/2}\left(\frac{1}{\kappa}\ \ln y^+ + C_2\right)\right]$$

(cf Rotta 1968, equation 17).

The above formula relates the compressible heat conducting velocity profile (in the generalized logarithmic region) to the incompressible adiabatic velocity profile (in the ordinary logarithmic region). If we let

$$U^+_{inc}(y^+) \equiv \frac{1}{\kappa} \ln y^+ + C_2$$

(in the ordinary logarithmic region), then the above compressible-incompressible transformation formula becomes

$$\frac{U}{u_\tau} = \frac{\beta_q}{(\gamma-1)M_\tau^{\,2}} + C_1^{\,1/2}\left(\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t\right)^{-1/2} \sin\left[\left(\frac{(\gamma-1)}{2}\,M_\tau^{\,2}\sigma_t\right)^{1/2} U^+_{inc}\,(y^+)\right]$$

$$- \frac{\beta_q}{(\gamma-1)M_\tau^{\,2}} \cos\left[\left(\frac{(\gamma-1)}{2}M_\tau^{\,2}\sigma_t\right)^{1/2} U^+_{inc}\,(y^+)\right] \qquad (5)$$

C. Von Kármán Momentum Integral

The third general result which we will employ in our boundary layer calculation algorithm is the Von Kármán momentum integral. The specific form which we wish to employ has been derived by many authors and, unlike the Rotta energy equation and compressibile-incompressible transformation formula, we feel no need to supply a derivation of it here. Readers are referred, instead, to the derivation of Young (1953).

Let the subscript δ denote conditions at the outer edge of the boundary layer. We define the displacement thickness δ^* by

$$\delta^* = \int_o^\delta \left(1 - \frac{\bar{\rho}(y)}{\bar{\rho}_\delta}\,\frac{U(y)}{U_\delta}\right) dy$$

We define the momentum thickness θ by

$$\theta = \int_o^\delta \frac{\bar{\rho}(y)}{\rho_\delta}\,\frac{U(y)}{U_\delta}\left(1 - \frac{U(y)}{U_\delta}\right) dy$$

Then the Von Kármán momentum integral is

$$\frac{d}{dx}\,(\bar{\rho}_\delta U_\delta^{\,2}\theta) = \delta^* \frac{dP}{dx} + \tau_w$$

with a variety of error terms among which are the apparent stresses at the outer edge of the boundary layer due to accoustic radiation effects and the effects of x-derivatives of the Reynolds normal stresses.

For flat plate flow, $dP/dx = 0$. Dividing by $\bar{\rho}_\delta U_\delta^2$, we obtain

$$\frac{d\theta}{dx} = \frac{\tau_w}{\bar{\rho}_\delta U_w^2} = \frac{\bar{\rho}_w u_\tau^2}{\bar{\rho}_\delta U_\delta^2}$$

If we recall (from the equation of state for an ideal gas and constancy of the pressure P) that $\bar{\rho}_w/\bar{\rho}_\delta = T_\delta/T_w$ and if we denote the free stream sound speed and Mach number by a_δ and M_δ, respectively, then

$$\frac{d\theta}{dx} = \frac{T_\delta}{T_w}\frac{a_w^2 M_\tau^2}{a_\delta^2 M_\delta^2} = \frac{T_\delta}{T_w}\frac{\gamma R T_w}{\gamma R T_\delta}\frac{M_\tau^2}{M_\delta^2}$$

or

$$\frac{d\theta}{dx} = \left(\frac{M_\tau}{M_\delta}\right)^2 \tag{6}$$

D. Consolodated Algorithm for Smooth and Rough Walls

For flat plate flow, M_δ is a constant. Inspection of equation (6) shows that if θ were a function of M_τ only, then one could eliminate either θ or M_τ form (6) and obtain a first order ordinary differential equation for whichever variable was not eliminated. Although it is hardly apparent at this stage, we will show in the present subsection that θ is indeed a function of M_τ only provided representative data is supplied that would normally constitute the input to a boundary layer calculation.

(a) Whole-Layer Formula for U^+_{inc} (y^+,k^+)

We begin by writing down two analytical curve fits to well known experimental data *in incompressible* flows which are usually presented in either graphical or tabular form.

Coles (1956, 1968) has shown that nearly all turbulent boundary layers (over smooth walls) can be fit to a formula of the form

$$U^+_{inc}(y^+) = f(y^+) + \frac{\Pi}{\kappa}\, w\!\left(\frac{y}{\delta}\right)$$

(smooth)

in which f and w are ostensibly universal functions determined experimentally and Π is a parameter (called "Coles' wake function"). A tabulation of $f(y^+)$ was given by Coles (1955). A tabulation of $w(y/\delta)$ was given by Coles (1956). An analytical curve fit to $w(y/\delta)$ was given by Coles (1968) as

$$w(\frac{y}{\delta}) = 2 \sin^2(\frac{\pi y}{2\delta}) = 1-\cos(\frac{\pi y}{\delta})$$

In the "logarithmic region," $f(y^+)$ has the well known form

$$f(y^+) = \frac{1}{\kappa} \ln y^+ + C_2 \tag{7}$$

In the sublayer and buffer region (say $y^+ < 35$) the above formula for $f(y^+)$ fails. The true law of the wall function must satisfy the constraint

$$\left(\frac{df}{dy^+}\right)_{y^+=0} = 0 \tag{8}$$

Now the functions tanh ( ) and $\sinh^{-1}($ $)$ are both "concave downward for all positive argument and have zero concavity at the origin--both properties of the law of the wall function $f(y^+)$. Also, any linear combination of $\sinh^{-1}($ $)$ and tanh ( ) will be asymptotic to a logarithm for large argument. An obvious curve fit to $f(y^+)$ is therefore a function of the form

$$f(y^+) = \frac{1}{\kappa} \sinh^{-1}(\frac{y^+}{2a}) + d \tanh(\frac{y^+}{c})$$

in which the parameters a, d, and c are constrained so that $f(y^+)$ is asymptotic to the form (7) for large y^+ and satisfies the slope constraint (8). We find (for smooth walls) that

$$d = C_2 + \frac{\ln a}{\kappa}$$

$$c = d(1 - \frac{1}{2\kappa a})^{-1}$$

and only "a" remains adjustable. By trial and error, we have found that

$$a = 2.45$$

leads to a uniformly valid approximation to Coles (1955) tabulation (assuming the values of κ and C_2 used there) with a maximum error under two percent.

Combining this fit to (y^+) with Coles (1968) fit to $w(y/\delta)$, a whole-layer formula for $U^+_{inc}(y^+)$ (for smooth walls) is found to be

$$\underset{(smooth)}{U^+_{inc}}(y^+) = \frac{1}{\kappa}\sinh^{-1}\left(\frac{y^+}{2a}\right) + d\tanh\left(\frac{y^+}{c}\right) + \frac{\Pi}{\kappa}\left(1-\cos\left(\frac{\pi y}{\delta}\right)\right)$$

with the above values of a,b, and c.

As a means of incorporating roughness effects, we will apply the well-known sand grain roughness data of Nikuradse (cf Cebeci and Bradshaw (1977), section 6.5). Let k now denote the roughness height (not to be confused with the thermal conductivity symbol used in subsection A above). Let

$$k^+ = \frac{ku_\tau}{\nu_w}$$

For a given roughness type, the velocity profile of a turbulent boundary layer exhibits a logarithmic region as on smooth walls, however the value of the additive constant C_2 in (7) is reduced by an amount $\Delta u^+(k^+)$. An analytical curve fit which agrees with Nikuradse's experimental points for $\Delta u^+(k^+)$ to within the scatter of those data is

$$\Delta u^+(k^+) = \frac{1}{2\kappa}\sinh^{-1}\left[\frac{1}{2}\left(\frac{k^+}{L_s^+}\right)^2\right] + \left(C_2 - B_{2\infty} + \frac{1}{\kappa}\ln L_s^+\right)\left[\frac{(k^+)^2}{(L_s^+)^2 + (k^+)^2}\right] \qquad (9)$$

with

$$(C_2,\ B_{2\infty},\ \kappa,\ L_s^+) = (5.5,\ 8.5,\ 0.40,\ 13.0)$$

For readers wishing to test this formula, we note that the Nikuradse data actually plotted in figure 6.16 of Cebeci and Bradshaw (1977) is the function $B_2(k^+)$ defined by

$$B_2(k^+) = \frac{1}{\kappa}\ln(k^+) + C_2 - \Delta u^+(k^+)$$

Since the main effect of sand grain roughness in the logarithmic part of the velocity profile is to reduce the additive constant C_2 by a roughness dependent amount $\Delta u^+(k^+)$, it seems most natural to incorporate roughness effects into our whole-layer velocity profile formula by an identical reduction of C_2 at the point where C_2 appears in that formula. Specifically, we may propose a whole-layer velocity profile formula of the form

$$U^+_{inc}(y^+,k^+) = \frac{1}{\kappa}\sinh^{-1}\left(\frac{y^+}{2a}\right) + d(k^+)\tanh\left(\frac{y^+}{c}\right) + \frac{\Pi}{\kappa}\left(1-\cos\left(\frac{\pi y}{\delta}\right)\right) \qquad (10a)$$

where

$$d(k^+) = C_2 - \Delta u^+(k^+) + \frac{\ln a}{\kappa} \qquad (10b)$$

and $\Delta u^+(k^+)$ is given by (9) above.

We will assume, for lack of better information, that c and a are not roughness dependent, i.e., a = 2.45 as before and

$$c = d_{smooth}\left(1-\frac{1}{2\kappa a}\right)^{-1} \qquad (10c)$$

where

$$d_{smooth} = C_2 + \frac{\ln a}{\kappa} \qquad (10d)$$

The slope of the velocity profile at the wall is of some interest. A simple calculation from the above formula shows that

$$\begin{aligned}\frac{\partial U^+_{inc}}{\partial y^+}(0,k^+) &= \frac{1}{2\kappa a} + \frac{d_{smooth}-\Delta u^+(k^+)}{c}\\ &= 1 - \frac{\Delta u^+}{c}\end{aligned}$$

which is positive if k^+ is between zero and 1,847.

(b) Thermal Boundary Conditions

In the subsection titled "Van Driest Transformation Theory" above we derived the formula

$$\frac{\bar{T}}{T_w} = C_1 + \beta_q \sigma_t \left(\frac{U}{u_\tau}\right) - \frac{M_\tau^2}{2} (\gamma - 1) \sigma_t \left(\frac{U}{u_\tau}\right)^2$$

Employing the subscript δ to denote conditions at the "outer edge" $y = \delta$ as before and employing the identity

$$\frac{U_\delta}{u_\tau} = \frac{M_\delta a_\delta}{M_\tau a_w} = \frac{M_\delta}{M_\tau} \left(\frac{T_\delta}{T_w}\right)^{1/2}$$

we have

$$\frac{T_\delta}{T_w} = C_1 + \beta_q \sigma_t \frac{M_\delta}{M_\tau} \left(\frac{T_\delta}{T_w}\right)^{1/2} - \frac{(\gamma-1)}{2} \sigma_t M_\delta^2 \frac{T_\delta}{T_w}$$

We will assume that the given data constituting the input to the boundary layer calculation will include a thermal boundary condition which will be either an adiabatic wall condition $\beta_q = 0$ or a given wall temperature condition T_w = given.

In the **adiabatic wall case** ($\beta_q=0$), the above formula yields a wall temperature formula of the form

$$(T_w)_{\beta_q=0} = \frac{T_\delta}{C_1} \left(1 + \frac{\gamma-1}{2} \sigma_t M_\delta^2\right) \tag{11a}$$

In the **given wall temperature case,** the above general formula may be solved for β_q:

$$\beta_q = \left(\sigma_t \frac{M_\delta}{M_\tau}\right)^{-1} \left(\frac{T_\delta}{T_w}\right)^{-1/2} \left[\frac{T_\delta}{T_w}\left(1 + \frac{\gamma-1}{2} \sigma_t M_\delta^2\right) - C_1\right] \tag{11b}$$

All of the parameters in (11b) other than M_τ would normally be known before the boundary layer calculation is started. In this sense (11b) defines a function $\beta_q(M_\tau)$ for given T_w. If β_q is given to be zero then (11a) defines T_w. In any case the quantities β_q and T_w are each either known from given data or expressible as an explicit function of M_τ.

(c) Determination of the Functions $\theta(M_\tau)$ and $M_\tau(x)$

In formula (1) of subsection B above, we introduced a variable ϕ which is a function of U/u_τ. Letting ϕ_δ denote the value of ϕ at $y = \delta$ and employing the now familiar identity

$$\frac{U_\delta}{u_\tau} = \frac{M_\delta}{M_\tau}\left(\frac{T_\delta}{T_w}\right)^{1/2},$$

we find from (1) that

$$\phi_\delta = \sin^{-1}\left\{\left[\left(\frac{\beta_q}{(\gamma-1)M_\tau^2}\right)^2 + \frac{2C_1}{(\gamma-1)M_\tau^2\sigma_t}\right]^{-1/2}\left[\frac{M_\delta}{M_\tau}\left(\frac{T_\delta}{T_w}\right)^{1/2} - \frac{\beta_q}{(\gamma-1)M_\tau^2}\right]\right\} \quad (12)$$

In that same subsection a constant of integration ϕ_o (which first appeared in formula (31)) arose and was assigned a specific value as a result of the requirement that the compressible heat conducting form of the law of the wall in the generalized logarithmic region reduce to the known incompressible adiabatic law of the wall as β_q and M_τ both tend to zero. Taking the arcsine of the formula for $\sin\phi_o$ so determined, we have

$$\phi_o = \sin^{-1}\left\{\left[\left(\frac{\beta_q}{(\gamma-1)M_\tau^2}\right)^2 + \frac{2C_1}{(\gamma-1)M_\tau^2\sigma_t}\right]^{-1/2}\left[\frac{-\beta_q}{(\gamma-1)M_\tau^2}\right]\right\} \quad (13)$$

The parameters ϕ_δ and ϕ_o are now known functions of M_τ once the thermal boundary conditions and the normal free stream data of the boundary layer claculation problem are specified.

Now we may obtain a whole-layer generalization of formula (3) of subsection B above by replacing the quanitity

$$\frac{1}{\kappa}\ln y^+ + C_2$$

by the more general expression $U^+_{inc}(y^+,k^+)$ as defined by forumla (10) of subsection D. In particular, at the outer edge $y = \delta$, we get

$$\phi_\delta - \phi_o = \left(\frac{(\gamma-1)}{2} M_\tau^2 \sigma_t\right)^{1/2} U^+_{inc}(\delta^+,k^+)$$

where, from (10),

$$U^+_{inc}(\delta^+,k^+) = \frac{1}{\kappa}\ln\delta^+ + C_2 - \Delta u^+(k^+) + \frac{2\Pi}{\kappa}$$

Eliminating U^+_{inc} between the last two equations and solving for δ^+, we get

$$\delta^+ = \exp\{\kappa[(\frac{\gamma-1}{2} M_\tau^{\,2}\sigma_t)^{-1/2}(\phi_\delta - \phi_o) - C_2 + \Delta u^+(k^+) - \frac{2\Pi}{\kappa}]\}$$

In view of previously derived results, the right hand side is a known function of M_τ provided the usual free stream data and thermal boundary conditions for a flat plate boundary layer calculation problem are specified.

The function $\theta(M_\tau)$ is now almost completely determined. The only remaining function we require is the function $l_{visc}(M_\tau)$ defined by

$$l_{visc}(M_\tau) = \frac{\nu_w}{u_\tau} = \frac{\mu(T_w)}{\bar{\rho}_w a_w M_\tau} = \frac{RT_w}{\bar{\rho}_w RT_w} \frac{\mu(T_w)}{(\gamma RT_w)^{1/2} M_\tau}$$

$$= \frac{1}{P}\left(\frac{RT_w}{\gamma}\right)^{1/2} \frac{\mu(T_w)}{M_\tau}$$

where P is the pressue and $\mu(T_w)$ is the wall value of the molecular viscosity $\mu(T)$. From the "Southerland law", we have

$$\mu(T) = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^{3/2} \frac{T_{ref} + S_1}{T + S_1}$$

where (for air)

$$\mu_{ref} = (0.350)\ 10^{-6}\ \frac{lb \cdot sec}{ft^2}$$

$T_{ref} = 492\ °R$
$S_1 = 198\ °R$

From the definition of the momentum thickness

$$\theta = \int_o^\delta \frac{\bar{\rho}(y)}{\rho_\delta} \frac{U(y)}{U_\delta}\left(1 - \frac{U(y)}{U_\delta}\right) dy$$

and the identities

$$\frac{\bar{\rho}(y)}{\rho_\delta} = \frac{T_\delta}{\bar{T}} = \frac{T_\delta}{T_w} \frac{T_w}{\bar{T}}$$

$$= \frac{T_\delta}{T_w} [C_1 + \beta_q \sigma_t \frac{U}{u_\tau} - \frac{M_\tau^2}{2} (\gamma-1)\sigma_t (\frac{U}{u_\tau})^2]^{-1} ,$$

$$\frac{U}{U_\delta} = \frac{u_\tau}{U_\delta} \frac{U}{u_\tau} = \frac{M_\tau}{M_\delta} (\frac{T_w}{T_\delta})^{1/2} \frac{U}{u_\tau} ,$$

and $y^+ = y/l_{visc}$, we have

$$\theta(M_\tau) = l_{visc} (M_\tau) \int_0^{\delta^+(M_\tau)} \{\frac{M_\tau}{M_\delta} (\frac{T_\delta}{T_w})^{1/2} [C_1 + \beta_q \sigma_t \frac{U}{u_\tau} - \frac{M_\tau^2}{2} (\gamma-1)\sigma_t (\frac{U}{u_\tau})^2]^{-1}$$

$$\cdot \frac{U}{u_\tau} [1 - \frac{M_\tau}{M_\delta} (\frac{T_w}{T_\delta})^{1/2} \frac{U}{u_\tau}]\} dy^+$$

Given the type of input data necessary for boundary layer calculation problems including free stream conditions and thermal boundary conditions and given the functions of M_τ already defined, the right hand side of the above formula for $\theta(M_\tau)$ is fully determined.

In particular the function $d\theta/dM_\tau$ is a known function of M_τ (though its evaluation would normally require numerical differentiation). It follows that the Von Kârmân momentum integral (cf. section C above)

$$\frac{d\theta}{dx} = (\frac{M_\tau}{M_\delta})^2$$

may be divided by $d\theta/dM_\tau$ to give

$$\frac{dM_\tau}{dx} = (\frac{M_\tau}{M_\delta})^2 [\frac{d\theta}{dM_\tau} (M_\tau)]^{-1}$$

which is a first order autonomous ordinary differential equation for M_τ as a function of x. It is only necessary to specify M_τ at an initial value of x to determine uniquely the function $M_\tau(x)$ for all downstream values of x.

IV. CALCULATION EXAMPLE

In this section, we will apply the algorithm developed in the preceeding section to a phsyical problem intended to model a boundary layer experiment currently

underway at the High Speed Aero Performance Branch of the Air Force Wright Aeronautical Laboratory in Dayton.

A. Physical Setup

Consider a supersonic wind tunnel with air as the working fluid. The tunnel reservoir conditions are denoted by the subscript "o". T_o and P_o are both given. From the design of the tunnel throat and test section, the test section Mach number M_δ is controllable and is therefore taken as a given quantity. Assuming the flow between the reservoir and the test section is isentropic, we have

$$T_\delta = T_o\left[1 + \frac{\gamma-1}{2} M_\delta^{\,2}\right]^{-1}$$

$$P_\delta = P_o \left(\frac{T_o}{T_\delta}\right)^{-\gamma/(\gamma-1)}$$

which determines the "free stream" temperature and pressue in the test section in terms of the given data.

We will assume that the model consists of a roughened flat plate at zero incidence. The leading edge of the plate coincides with the straight line x=o, y=o in the (x,y,z) coordinate system. The free stream velocity is in the direction of the positive x axis.

The plate is smooth in the strip $0 < x <$ immediately behind the leading edge. Behind that strip the plate has a uniformly distributed roughness of height k.

B. The Starting Laminar Boundary Layer

The development of a flat plate laminar boundary layer in a compressible flow can be calculated analytically (cf. Schlichting (1968), Chapter XIII) provided the viscosity-temperature relation has the idealized form

$$\mu(T) = \mu(T_{ref}) \left(\frac{T}{T_{ref}}\right)^{\omega}$$

where ω and T_{ref} are constants. Notice that the above formula implies

$$\frac{\mu(T_\delta)}{\mu(T_w)} = (T_\delta/T_w)^\omega$$

If T_δ is given and T_w is determined from the thermal boundary conditions of the problem then $\mu(T_\delta)$ and $\mu(T_w)$ can be calculated from the Southerland law. The above formula then determines ω.

For example, if T_o = 1100 °R and M_δ = 6 then from the isentropic relations T_δ = 134.15 °R. If the wall is adiabatic and the "turbulent Prandtl number" is taken to be σ_t = 0.9 (Rotta (1960), then from the formula for T_w in section D(b) above, we get T_w = 1003 °R. From the Southerland law, the values

$$\mu(T_\delta) = (1.035)\ 10^{-7}\ \text{lb} \cdot \text{sec/ft}^2$$

$$\mu(T_w) = (6.601)\ 10^{-7}\ \text{lb} \cdot \text{sec/ft}^2$$

follow. The exponent ω is then ω = 0.8611.

For this value of ω and M_δ = 6, figure 13.8 of Schlichting (1968) gives, approximately,

$$\frac{\tau_w}{\frac{\bar{\rho}_\delta U_\delta^2}{2}} = 1.16\ [\mu(T_\delta)/(\bar{\rho}_\delta U_\delta x)]^{-1/2}$$

But
$$\frac{\tau_w}{\frac{\bar{\rho}_\delta U_\delta^2}{2}} = 2\,\frac{\bar{\rho}_w}{\bar{\rho}_\delta}\,\frac{u_\tau^2}{U_\delta^2} = 2\,\frac{\bar{\rho}_w}{\bar{\rho}_\delta}\,\frac{M_\tau^2}{M_\delta^2}\,\frac{T_w}{T_\delta} = 2\,\frac{M_\tau^2}{M_\delta^2}$$

so the parameter M_τ for laminar (adiabatic) flow is expressed in terms of M_δ and the Reynolds number based on x.

For example, if l = 1.0 inch then the values of M_τ corresponding to P_o = (700, 1400, 2100) psia T_o = 1100 °R are M_τ = (0.1547, 0.1301, 0.1176). If laminar-turbulent transition is assumed to take place within an infinitesimally narrow x-interval corresponding to the start of the rough part of the plate, then the starting conditions for the solution of the first order ordinary differential

equation for $M_\tau(x)$ given at the end of section VI above may be defined such that the starting M_τ at the begining of the turbulent boundary layer correspond to the same value of the momentum thickness as at the end of the laminar region.

C. Specific Calculations

The constants of intergration C_1 and C_2 (which were introduced in section III B above) arose during the evaluation of a y integral. Accordingly, they are independent of y. They may however depend on the parameters M_τ and β_q. Bradshaw (1977) suggests the functions

$$C_1 = 1.0$$

$$C_2 = 5.0 + 95.0\, M_\tau^{\,2} + 30.7\, \beta_q + 226.0\, \beta_q^{\,2}$$

A few other constants not yet specified were used in the calculations. They are

(i) The Kármán constant: $\kappa = 0.41$
(ii) The ratio of specific heats for air: $\gamma = 1.4$
(iii) The gas constant for air: $R = 1716.48$ (ft. lbs)/(slug °R)
(iv) The Coles wake parameter for flat plate flow: $\Pi = 0.62$

The derivative with respect to M_τ of the function $\theta(M_\tau)$ was approximated numerically by a formula of the form

$$\frac{d\theta}{dM_\tau} \approx \frac{\theta[(1+\varepsilon)M_\tau] - \theta[(1-\varepsilon)M_\tau]}{2\,\varepsilon\, M_\tau}$$

with $\varepsilon = 0.05$.

Figures 1 and 2 illustrate the effect of surface roughness height on the distributions of skin friction and momentum thickness, respectively for adiabatic wall conditions. Figures 3, 4, and 5 illustrate the effect of wall temperature on the distributions of skin friction parameter, all for a fixed roughness height k = 0.04 inches.

The calculation results of figures 1-5 were all based on the same assumptions regarding tunnel stagnation temperature and pressure. These were $T_o = 1100$ °R and $P_o = 1400$ psia, respectively.

The results agree with the trends that one might reasonably expect. The only thing that might be surprising to readers not intimately familiar with high speed boundary layers is the small numerical value of the momentum thickness. This can be rationalized to some extent by noting that the absolute temperature of the fluid on the wall is about nine times that of the fluid at the outer edge. Thus the mass density of the fluid on the wall is about one ninth that of the fluid in the free stream. From the definition of the momentum thickness

$$\theta = \int_o^\delta \frac{\bar{\rho}(y)}{\rho_\delta} \frac{U(y)}{U_\delta} \left(1 - \frac{U(y)}{U_\delta}\right) dy$$

one sees that a small value of $\bar{\rho}(y)$ in the region where U/U_δ is intermediate between its extremes tends to make the value of θ smaller than it would be for uniform density.

V. CONCLUSIONS AND RECOMMENDATIONS

Despite the crudeness of some of the assumptions employed in the derivation of the above algorithm (such as, for example, the neglect of energy losses due to acoustic radiation and the wave drag of individual roughness elements) the results appear quite plausible, and suggest that simple algorithms may be quite adequate for the prediction of statistically two dimensional flat plate boundary layers.

We feel that our objective of furnishing a self-contained derivation of an efficient and physically reasonable boundary layer prediction method has been achieved. Further work should, however, address the neglected physical effects mentioned in the preceeding paragraph.

Norman, Oklahoma
June 30, 1984

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```
      program tblcom
      integer adwall,n,i,j,m,k,done,steps,itratr,
     -   maxstp
      real p0,t0,mdelta,gamma,pnturb,betaq(1:41),
     - coles,karman,lrough,awall,c1,muref,
     - tref,s1,krough,rgas,mu,c2,t0ovtd,
     - pdelta,rhodel,tdelta,tw,betaq,mtau(1:41),
     - mtau1,f,theta(1:41),lvisc,kplus,dolpls,mtau2,
     - phidel,phi0,uplus,incupl,d,cinv,
     - esmall,count,f112,f1221,temp,
     - asinh,x,kfunc,chmtau,awall,group1,group2,
     - group3,upldel,z(1:6),w(1:6),ypl,duruf,
     - yplbot,damoo,ypltop,pi,denom,
     - ntgran,ntgrl,theta1,crk,h,dtheta,
     - udelta,momlam,xlam
      implicit logical(a-z)
      data t0,   mdelta,gamma,pnturb,coles,
     -    1100.,    6.,   1.4  , .9  , .62/,
     -    karman,lrough,awall,  c1,muref/
     -     .41  ,  13  ,2.45 ,  1. ,.000000350/,
     -    tref,   s1, krough, rgas /
     -    492.,  198.,.00167,1716.48/,
     -    z(1)  , z(3),  z(5),  w(1)/
     - .2386191861,.6612093865,.9324695142,
     - .4679139346/,w(3),w(5),pi/.360761573,.1713244924,
     - 3.141592654/,esmall/.1/
c
      f1221(count)=count*(5.-count)*.5-1.
      f112(count)=(count-2.)*(count-1.)*.5+1.
      c2(mtau1,betaq)=5.+95.*mtau1*mtau1
     -  +betaq*(30.7+226.*betaq)
      mu(temp)=muref*((temp/tref)**1.5)
     - *(tref+s1)/(temp+s1)
      asinh(x)=log(x+sqrt(1.+x*x))
      kfunc(kplus)=kplus*kplus/(lrough*lrough
      duruf(kplus)=.5*asinh(.5*kfunc(kplus))/karman
     - +(log(lrough)/karman-3.)*kfunc(kplus)
     - /(1.+kfunc(kplus))
      do 3 i=1,5,2
        z(i+1)=-z(i)
  3     w(i+1)=w(i)
c
  6   print*,'this program predicts the'
      print*,'downstream development of a
      print*,'flat plate turbulent boundary'
```

```
print*,'layer including effects of'
print*,'compressibility, roughness,'
print*,'and heat transfer.  to use it,'
print*.'the user must specify what'
print*,'type of thermal boundary '
print*,'condition applies'
t0ovtd=1.+.5*(gamma-1.)*mdelta*mdelta
tdelta=t0/t0ovtd
print*,'type 0 if the wall is adiabatic,'
print*,'type 1 if the wall in not'
print*,'adiabatic....'
read*, adwall
if(adwall.eq.0) then
betaq=0.
tw=tdelta*(1.+.5*(gamma-1.)*prturb*mdelta
 *mdelta)/c1
else
print*,'since you have indicated that'
print*,'the wall is nonadiabatic, a'
print*,'wall temperature is required.'
print*,'type it in (assuming degrees'
print*,' rankine)...'
read*, tw
endif
print*,'the current value in storage'
print*,'for the tunnel resevoir'
print*,'temperature is',t0,' degrees rankine'
print*,'the test section mach number is'
print*,mdelta
print*,'type in the tunnel resevoir pressure'
print*,'(in psia)...'
read*,p0
p0=p0*144.
pdelta=p0*t0ovtd**(gamma/(1.-gamma))
rhodel=pdelta*t0ovtd/(rgas*t0)
c
print*,'type in the initial value of the'
print*,'parameter mtau=mdelta*sqrt(cf/2)'
read*,mtau(1)
print*,'type in the integration step size
print*,'in inches...'
read*,h
h=h/12.
print*,'type in the number of steps'
print*,'over which output data is '
```

```
      print*,'required...'
      read*,n
      print*,'what is the equivalent sand'
      print*,'grain roughness height?(in'
      print*,'inches)?'
      read*,krough
      krough=krough/12.
      steps=2
      print*,'what is the length of'
      print*,'the initial laminar run
      print*,'(in inches)?'
      read*,xlam
      xlam=xlam/12.
      udelta=mdelta*sqrt(gamma*rgas*tdelta
      print*,'udelta=',udelta,' tdelta=',tdelta
      print*,' xlam=',xlam,' rhodel=',rhodel
      momlam=.58*sqrt(mu(tdelta)*xlam/(udelta
     -  *rhodel))
    7 print*,'the momentum thickness at'
      print*,'the end of the laminar run'
      print*,'(called momlam) is',momlam,'feet'
    8 continue
c
c begin (outer) runge-kutta loop
      do 2 i=1,steps
      count=1.
      mtau2=mtau(i)
      theta(i)=0.
      chmtau=0.
c begin calculation of the slope function f(mtau2)
c which is the right hand side of the first
c order ordinary differential equation to be
c solved
    1 obetaq(i)=0.
      dtheta=0.
      do 5 m=-1,1,2
      mtau1=mtau2*(1.+esmall*real(m))
      upldel=mdelta*sqrt(tdelta/tw)/mtau1
      lvisc=sqrt(rgas*tw/gamma)*mu(tw)/(pdelta*mtau1
      kplus=krough/lvisc
      group1=(gamma-1.)*mtau1*mtau1
      group2=sqrt(.5*pnturb*group1)
      if(aowall.ne.0) then
      betaq=(tdelta*(1.+.5*(gamma-1.)*pnturb
     - *mdelta*mdelta)/tw-c1)/(pnturb*upldel)
```

```
      obetaq(i)=obetaq(i)+.5*betaq
      endif
      group3=sqrt(c1+.5*pnturb*betaq*betaq/group1)
      phidel=asin(group2*(upldel-betaq/group1)
     - /group3)
      phi0=asin(group2*(-betaq/group1)/group3)
      delpls=exp(karman*((phidel-phi0)/group2
     -  +duruf(kplus)-c2(mtqul,betaq))-2.*coles)
c
c begin gaussean quadrature to calculate
c momentum thickness
      if(delpls.le.80.) then
      print*,'dubious delpls.  delpls=',delpls
      print*,'m=',m,' i=',i
      print*,'count=',count,' kplus=',kplus
      print*,'group2=',group2
      endif
      ntgrl=0.
      dsmoo=c2(mtqul,betaq)+log(awall)/karman
      d=dsmoo-duruf(kplus)
      cinv=(1.-.5/(karman*awall))/dsmoo
      j=0
      edgebl=.false.
      yplbot=0.
      ypltop=25.
      maxstp=15
   10 continue
      do  4 k=1,6
      ypl=.5*(z(k)*(ypltop-yplbot)
     -   +ypltop+yplbot)
      incupl=d*tanh(ypl*cinv)+(asinh(.5*ypl/awall)
     - +coles*(1.-cos(pi*ypl/delpls)))/karman
      uplus=sqrt(c1)*sin(group2*incupl)/group2
     - +betaq*(1.-cos(group2*incupl))/group1
      denom=pnturb*uplus*(betaq+.5*group1*uplus)+c1
      ntgran=uplus*(1.-uplus/upldel)/denom
    4 ntgrl=ntgrl+.5*(ypltop-yplbot)*w(k)
     -   *ntgran
      yplbot=ypltop
      ypltop=ypltop*2.
      j=j+1
      if (j.gt.maxstp) then
      print*,'theta integral not'
      print*,'complete after',maxstp,
     -   ' steps.  quit calculation'
```

```
      i=steps
      go to 11
      elseif(edgebl.eq..true.) then
      go to 11
      endif
      if (ypltop.ge.delpls) then
      edgebl=.true.
      ypltop=delpls
      endif
      go to 10
 11   thetq1=lvisc*ntgrl*tdelta/(tw*upldel)
      if(count.eq.1.)then
        theta(i)=theta(i)+.5*theta1
      endif
  5   dtheta=dtheta+.5*real(m)*theta1/(esmall*mtau2)
      if (i.eq.steps) go to 2
      f=mtau2*mtau2/(mdelta*mdelta*dtheta)
c
      crk=h*f*.5
      chmtau=chmtau+f1221(count)*crk/3.
      if(count.eq.4.) go to 9
      mtau2=mtau(i)+f112(count)*crk
      count=count+1.
      go to 1
  9   continue
      mtau(i+1)=mtau(i)+chmtau
  2   continue
c
      print'(5x,a,10x,a,7x,a,7x,a)','x','mtau',
     - 'theta','betaq'
      print '(1x,4f12.9)',(h*real(i),mtau(i),
     - theta(i),obetaq(i),i=1,steps)
      if (steps.le.2) then
      print*,' type 1 if the first theta
      print*,'is close to momlqm.  type 0'
      print*,'if the agreement is poor'
      read*,itratr
         if(itratr.eq.0)then
      print*,'type in a better value of'
      print*,'mtau(1).....'
      read*,mtau(1)
          go to 7
         else
          steps=r
          go to 8
```

```
      endif
    endif
    if(qdwall.ne.0)then
    write(1,'(1x,a,f7.1)')'given tw=',tw
    endif
    write(1,'(1x,a,i2,a,f5.0,a,f7.5)')'qdwall=',
-    qdwall,'p0 (in psiq)=',p0/144.,'mtau(1)=',
-    mtau(1)
    write(1,'(1x,a,f5.4)')'krough=',krough*12
    write(1,'(5x,a,10x,a,7x,a,7x,a)')'x','mtau',
-    'theta','betaq'
    write(1,'(1x,4f12.9)')(h*real(i),mtau(i),
-    theta(i),obetaq(i),i=1,n)
    print*,'are you finished?(type 0'
    print*,'if you are, type 1 if you'
    print*,'want to do another example'
    read*,done
    if (done.ne.0) go to 6
    endfile(1)
    rewind(1)
    end
```

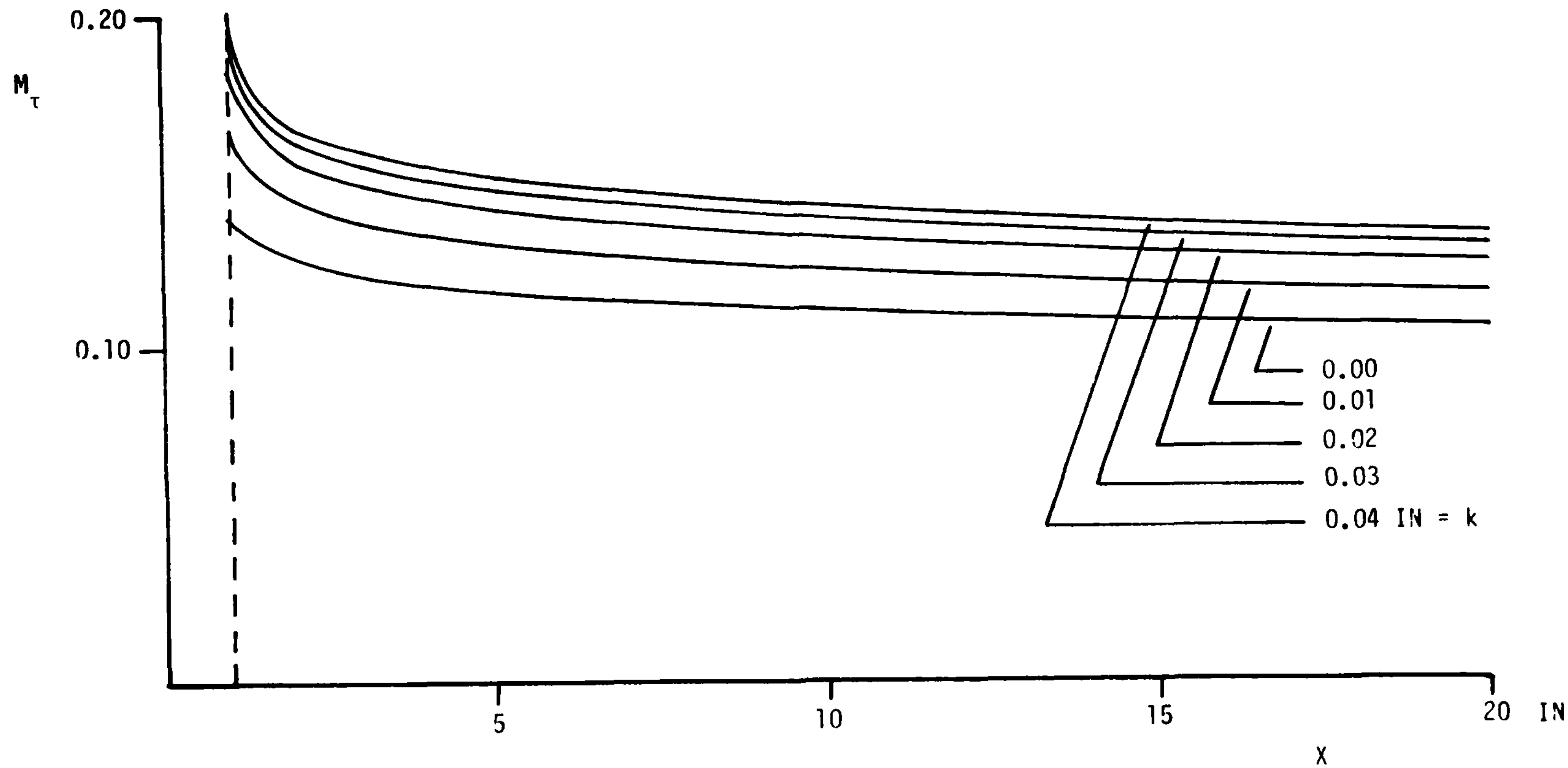


Figure 1 Skin friction parameter $M_\tau = M_\delta(c_f/2)^{1/2}$ versus x for several roughness heights. $M_\delta = 6.0$. Tunnel resevoir conditions: $T_o = 1{,}100\ ^oR$, $P_o = 1{,}400$ psia. Adiabatic wall: $T_w = 1{,}003\ ^oR$. Dashed line indicates start of the rough part of the plate.

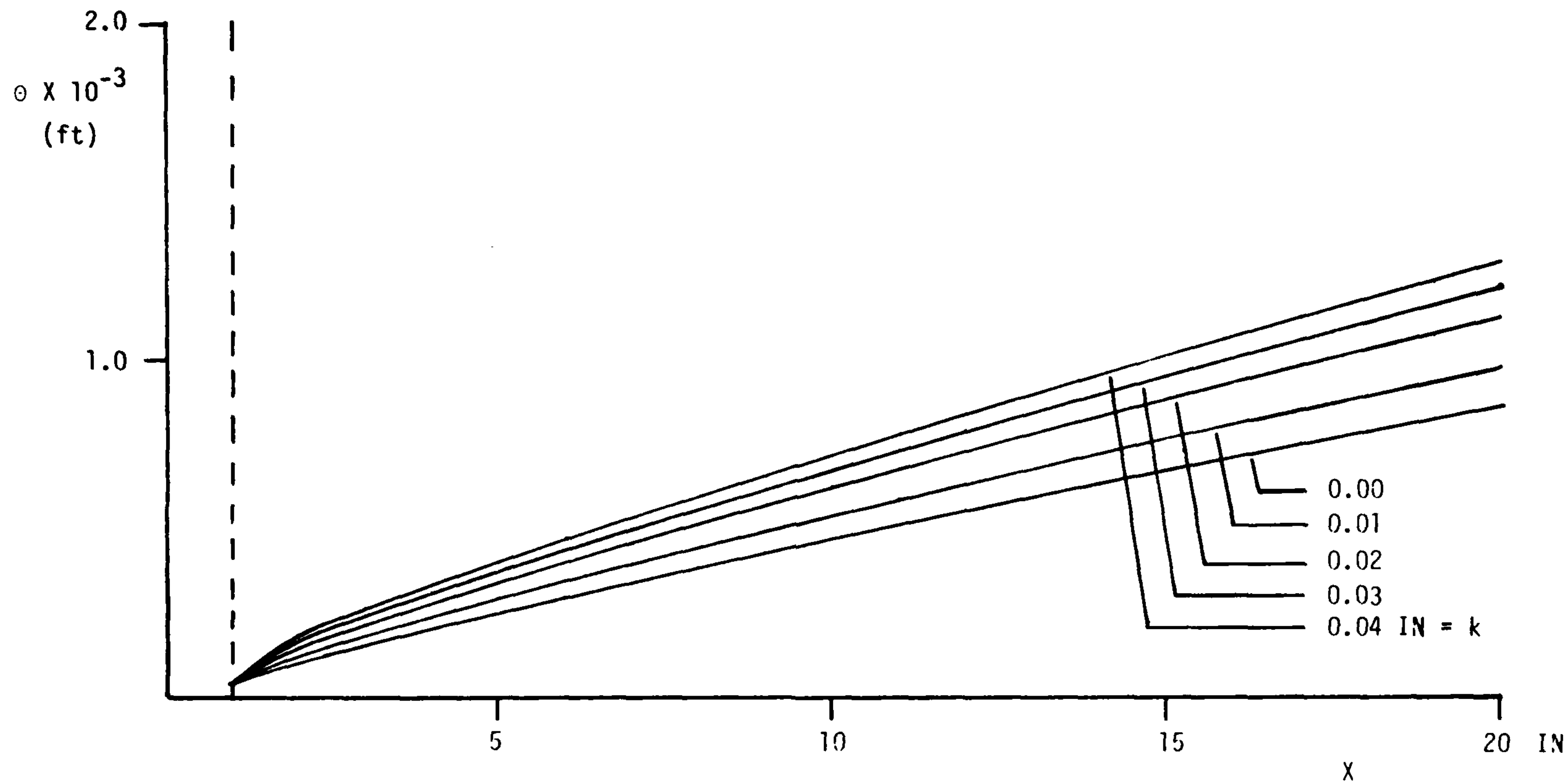
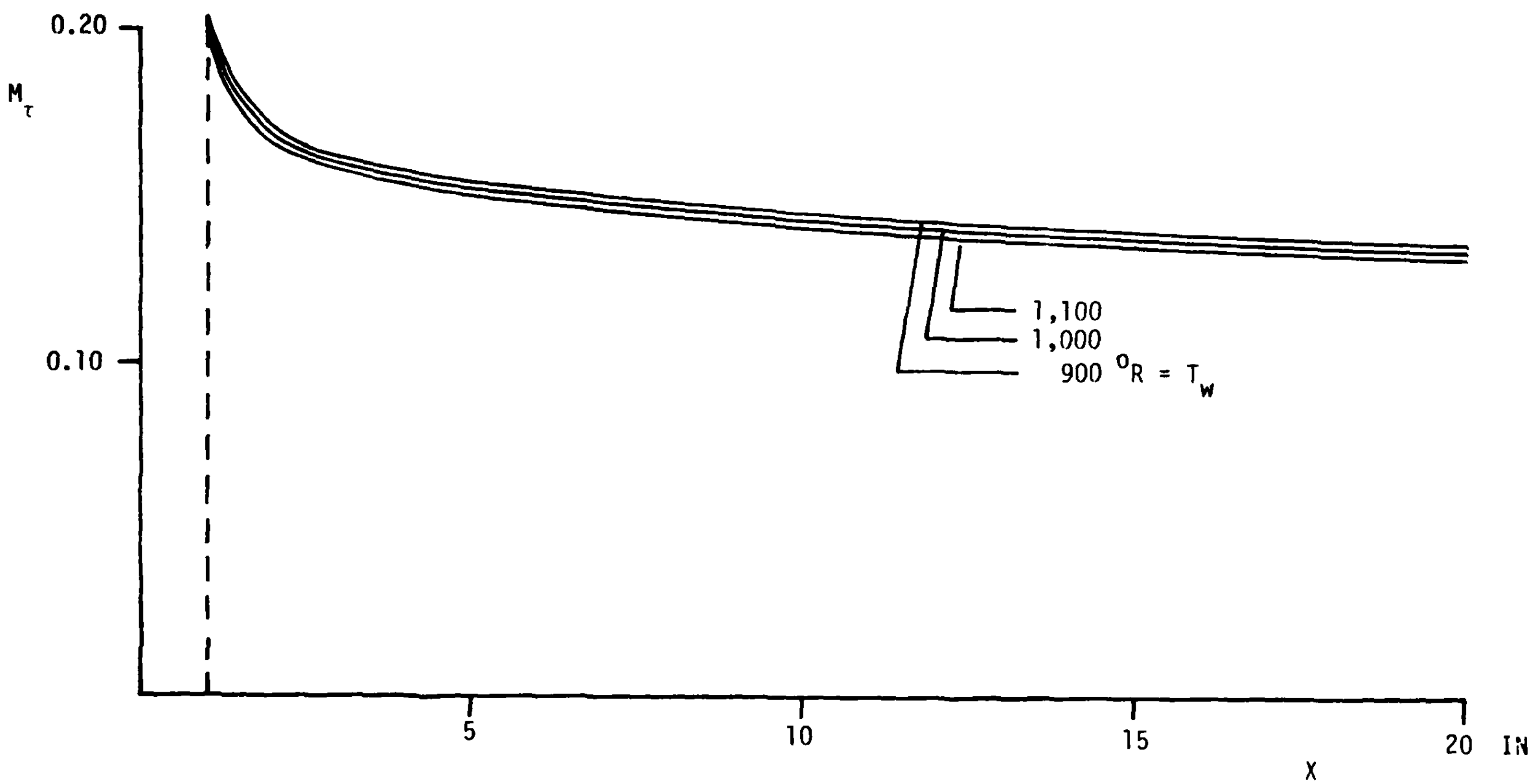


Figure 2 Momentum thickness versus x for several roughness heights. Adiabatic wall. Tunnel conditions same as in figure 1.



<u>Figure 3</u> Skin friction parameter M_τ versus x for fixed roughness height k = 0.04 in and several wall temperatures. Tunnel conditions same as in figure 1.

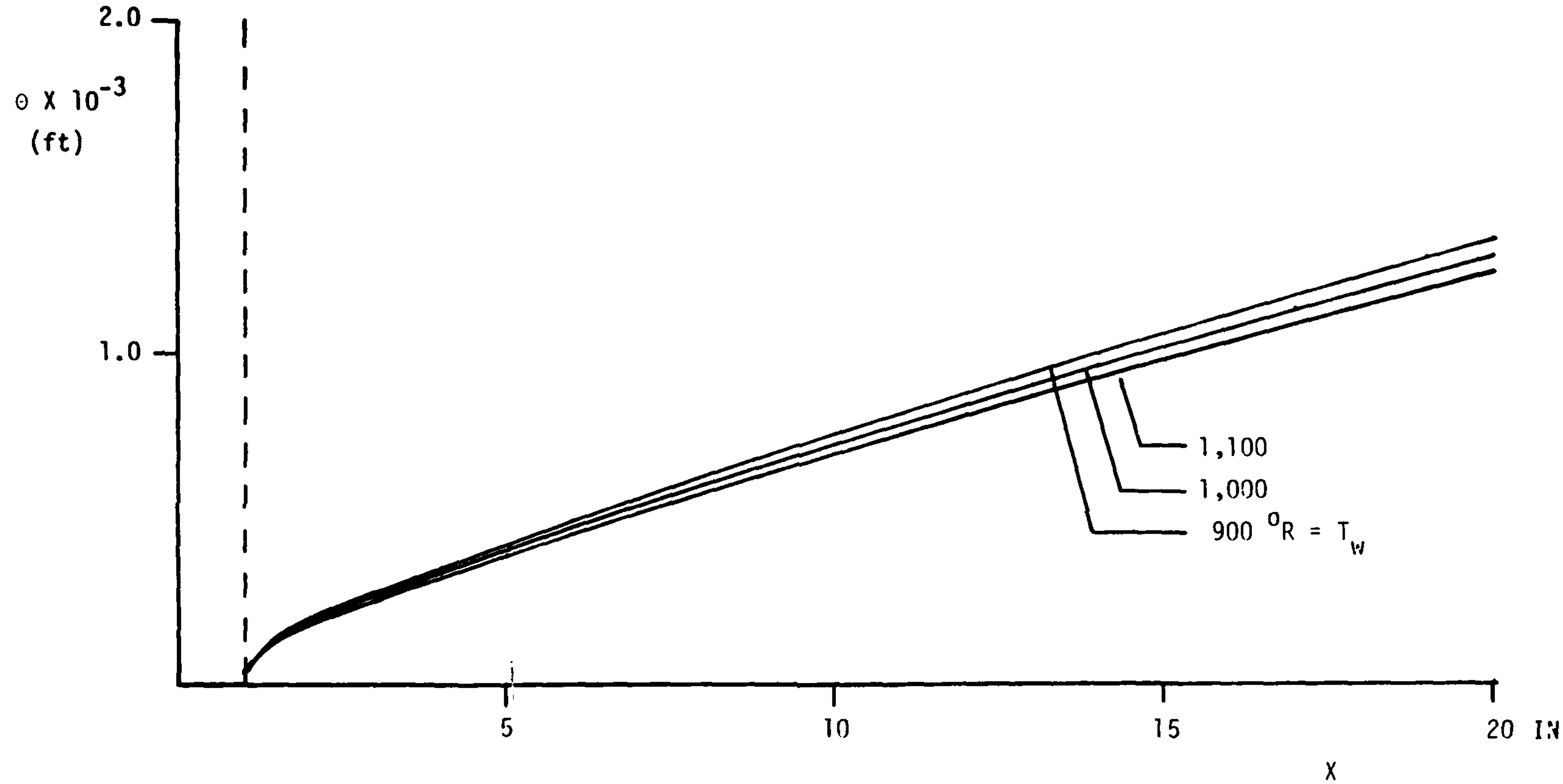
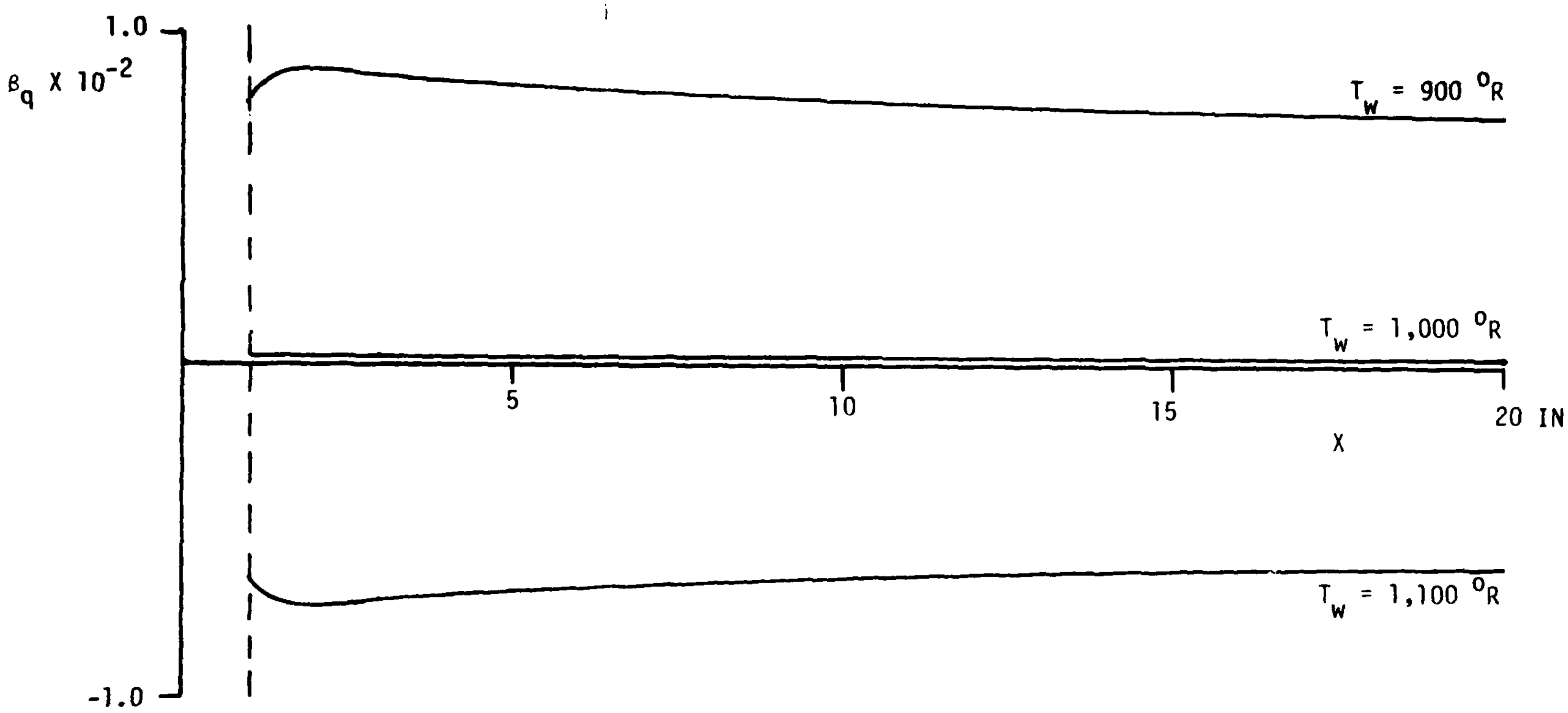


Figure 4 Momentum thickness versus x for fixed roughness height k = 0.04 in and several wall temperatures. Tunnel conditions same as in figure 1.



<u>Figure 5</u> Heat transfer parameter versus x for fixed roughness height k = 0.04 in and several wall temperatures. Tunnel conditions same as in figure 1.[$\beta_q \equiv \dot{q}_w / (\rho_w u_\tau T_w c_p)$]